If iteration is to be performed, the A, B, C, and D matrices are revaluated with the better values of the quantities at $(m + \frac{1}{2}, n)$ now available Then new values of w_n and $V_{m+1/2\ n}$ can be obtained Iteration may be performed at every step or not at all; in either case the foregoing procedure is applied at succeeding steps downstream until the boundary-layer flow along the body is determined

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Nondimensional Solutions of Flows with Vibrational Relaxation

RALPH PHINNEY*
Martin Marietta Corporation, Baltimore, Md

Vibrational relaxation time data for diatomic gases are reduced to a single curve by a suitable nondimensionalization Using this, together with a previously developed criterion for vibrational freezing, nozzle flow and a simplified blunt-body problem are solved in nondimensional form The effects of freezing on wind tunnel test section and model afterbody conditions are also presented

Nomenclature

A = area
D = nose diameter
E = vibrational energy
J = see Eq (7)
M = Mach no
p = pressure in atmospheres
R = gas constant
r_* = throat radius
T = temperature nondimensionalized with θ
t = time
u = velocity
θ = vibrational characteristic temperature
ρ = density
τ' = relaxation time at atmospheric pressure
τ = relaxation time, τ'/p
ϕ = asymptotic cone half angle of nozzle
ψ = $(2x/D)$, angle between body tangent and the normal to the freestream velocity

Subscripts

0 = stagnation
* = sonic
∞ = asymptotic, fully expanded flow
e = equilibrium flow
f = frozen flow
F = freeze point value

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* Research Scientist, Martin Space Systems Division

Introduction

In the flow of high-temperature gases, it is commonplace that the relaxation rates associated with various excited states and chemical reactions are not fast enough to permit the gas to remain in equilibrium A number of exact calculations of such flows, combining the flow equations and those for the relaxation, have been made (for example, Refs 1 and 2) These are long and tedious, and, in any case, their accuracy is limited by the usually rather unprecise knowledge of the rate constants Further, it is difficult to draw general conclusions or make parametric studies in this manner

In the expansion process of a nozzle or the flow around a blunt body, the flow of a relaxing gas proceeds from a high-temperature region where the gas is in equilibrium through a transition zone where it departs abruptly from equilibrium This occurs when the temperature and density fall so rapidly that the reaction rate quickly becomes so slow as not to permit continuation of an equilibrium state After the transi

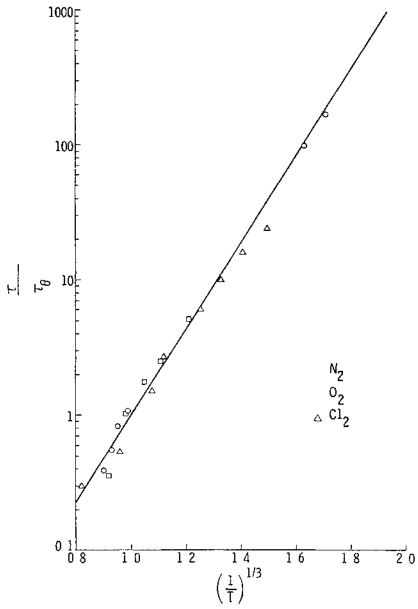

Fig 1 Nondimensional relaxation time data for diatomic gases

tion, the now cool gas is essentially frozen, the relaxation being so slow as to be negligible

Observation of these features has led to the "freezing point" approximation first introduced to study dissociational relaxation (Refs 3–5, for example) and later applied to vibrational relaxation (Refs 6 and 7) Here, the flow is considered to remain in equilibrium to some point at which freezing occurs and remains frozen from then on This greatly simplifies the analysis of relaxing flows The loss of precision in this method is more than compensated for by the generality of solutions In any case, such inaccuracies are probably no more severe than those due to the knowledge of the rate process

In the present paper, it will be shown first that vibrational relaxation data can be put into a convenient nondimensional form applicable to a variety of (diatomic) gases Then, using the freezing point criterion of Ref 7, the vibrationally relaxing flow in a hyperbolic nozzle and about a blunt body will be investigated

Analysis

Relaxation Time

Fairly extensive data on the temperature effect on relaxation time exists only for the three diatomic gases: nitrogen, oxygen, and chlorine This data is contained primarily in Refs 10–12, which are summarized together with some other scattered data in Ref 8

Let τ' be the relaxation time at atmospheric pressure and θ be the characteristic temperature of vibration of the gas From the data of Ref 8 for N_2, O_2, and Cl_2, one can determine the value of τ' at the temperature θ Call this time τ_θ' Values are given in Table 1 If all the forementioned data are then plotted in the coordinates τ'/τ_θ' vs $(1/T)^{1/3}$ (Fig 1), it is found that the data collapse to a single curve The straight line fitting the data of Ref 8 best is given by

$$\log_{10}\frac{\tau'}{\tau_\theta'} = 3\ 211\left[\left(\frac{1}{T}\right)^{1/3} - 1\right] \tag{1}$$

Equation (1) is in a form such that ratios of relaxation times are easily obtained without τ_θ' being involved explicitly Hence, the relaxation time can always be referred to the stagnation value with no knowledge of τ_θ' necessary As a result, Eq (1) is used only as an interpolation formula from τ_θ' and need be accurate only over the range considered in the particular problem instead of the full range including τ_θ'

Freeze Point Criterion

The sudden freezing of vibrational energy can be studied in exactly the same way as the freezing of dissociational energy In Ref 7, the freezing point is defined as the location where

$$E\ dE/dt = 1/\tau \tag{2}$$

E being the vibrational energy of the gas It was shown that this gives reasonable agreement with the freezing point as determined from step-by-step calculations of the relaxation process Reference 6 has a similar criterion which differs from Eq (2) by a factor that depends on temperature but is of order one Equation (2) will be used as the freezing point criterion throughout the rest of this analysis

Nozzle Flow

The general one-dimensional equations for equilibrium or for frozen flow are given in Ref 9 Taking T as the independent variable and using the subscript 0 to indicate stagnation values (assuming stagnation conditions to be in equilibrium), the equations for the thermodynamic variables u, ρ, and p are

$$\frac{u}{(2R\theta T_0)^{1/2}} = \left[\frac{7}{2}\left(1 - \frac{T}{T_0}\right) + \frac{E_0 - E}{R\theta T_0}\right]^{1/2} \tag{3}$$

$$\frac{\rho}{\rho_0} = \left(\frac{T}{T_0}\right)^{5/2}\frac{\exp J}{\exp J_0} \tag{4}$$

$$\frac{p}{p_0} = \left(\frac{T}{T_0}\right)^{7/2}\frac{\exp J}{\exp J_0} \tag{5}$$

where

$$E/R\theta = (e^{1/T} - 1)^{-1} \tag{6}$$

and

$$\exp J = \frac{E}{R\theta}\exp\left[\frac{1 + (E/R\theta)}{T}\right] \tag{7}$$

and T is the temperature nondimensionalized with respect to θ, p is the pressure in atmospheres, and the other variables can be in any consistent set of units Equations (3–5) apply for either equilibrium or frozen flows—in the frozen case, E is fixed at the freeze point value E_F, corresponding to the freeze point temperature T_F

With T as the independent variable, the criterion of Eq (2) can be factored into the following form:

$$\left(\frac{\tau}{E}\right)\left(\frac{dE}{dT}\right)\left(\frac{dT}{dA/A_*}\right)\left(\frac{dA/A_*}{dx}\right)\left(\frac{dx}{dt}\right) = 1 \tag{8}$$

For a hyperbolic nozzle with throat radius of r_* and an asymptotic cone angle of ϕ, the equation of the nozzle contour is

$$A/A_* = 1 + (x\tan\phi/r_*)^2 \tag{9}$$

so that Eq (8) can be written in the form

$$\left(\frac{p_0}{p}\frac{\tau'}{\tau_0'}\right)\left(\frac{R\theta}{E}\frac{dE/R\theta}{dT}\right)\left(\frac{dA/A_*}{dT}\right)^{-1}\times$$

$$\left(\frac{A}{A_*} - 1\right)^{1/2}\frac{u}{(2R\theta T_0)^{1/2}} = \frac{r_*}{2\tau_0\tan\phi\,(2R\theta T_0)^{1/2}} \tag{10}$$

where the right-hand side of Eq (10) is a dimensionless parameter that determines the extent of frozen flow If we assume the flow to be in equilibrium to the throat so that A_* is independent of the freezing point temperature, then

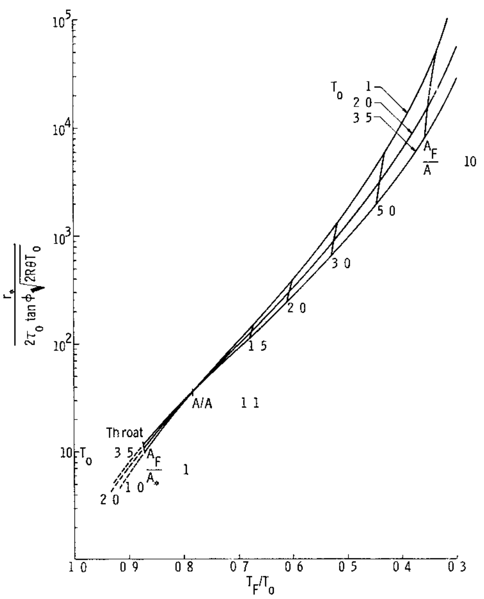

Fig 2 Vibration freezing point criterion for nozzle flow

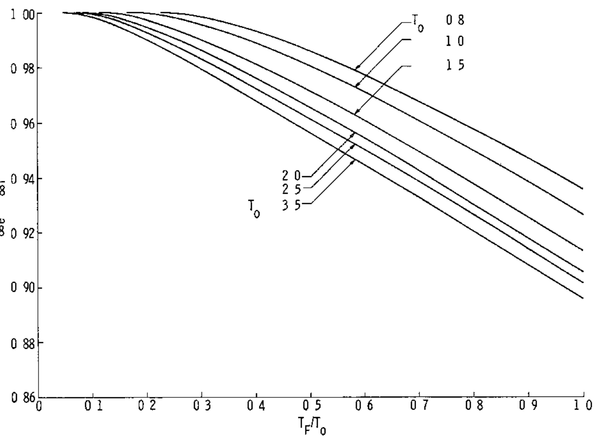

Fig 3 Effect of vibrational freezing on velocity and density in a nozzle

through the continuity equation A/A_* can be expressed in terms of u and ρ, which are both functions of T alone As a result, the left side of Eq (10) is a function of T only, with T_0 as a parameter

Figure 2 is a plot of the left side of Eq (10) In the absence of molecular vibration, this function would depend on T/T_0 only and not upon T_0: hence, the close similarity of the curves for different values of stagnation temperatures Figure 2 can be employed to find the freeze point by calculating the parameter $r_*/[2\tau_0 \tan\phi\ (2R\theta T_0)^{1/2}]$ and reading off the corresponding freeze point temperature T_F or area ratio A_F/A_* As will be seen in the next section, this definition of the freeze point (T_F) is most useful in determining the test section effects of freezing

Test Section Effects of "Frozen-Out" Energy

Consider the idealized case of nozzle flow in which the gas undergoes an equilibrium expansion to the freezing point (indicated by the subscript F) after which it expands with the vibrational energy remaining constant at the value E_F If the frozen flow is carried to the same area ratio (test section size) as the equilibrium flow, then we will get a pressure p_f instead of p_e for the equilibrium case When the area ratio is large enough, the velocity approaches its asymptotic value u_∞ or $u_{\infty f}$ and p_f/p approaches a limit that depends on the stagnation and freezing point temperatures

Table 1 Characteristic temperature of vibration and characteristic relaxation time

Gas	θ K	τ_θ' μsec
N_2	3380	18
O_2	2230	3 95
Cl_2	810	0 265

Over a considerable range, the velocity varies little from its asymptotic value, whereas pressure, density, and temperature change considerably, having no asymptotic values If, for example, the test section Mach number is larger than about 17, the velocity is within 1% of its maximum value, and if $M = 8$, it is within 5% In order to simplify the problem of comparison of frozen and equilibrium flow, it is advantageous to consider the asymptotic high Mach number case where the velocity is $u_{\infty f}$ for frozen flow, or u_∞ for equilibrium flow The ratio of pressures p_f/p will approach a limit $p_{\infty f}/p_\infty$ even though $p_{\infty f}$ and p_∞ do not separately approach limits

In Eq (3), if we let $T \to 0$, this corresponds to the maximum velocity obtainable for an infinite Mach number If E_F is the freeze value of the vibrational energy, then

$$u_{\infty f} = (2R\theta T_0)^{1/2}\left(\frac{7}{2} + \frac{E_0 - E_F}{R\theta T_0}\right)^{1/2} \tag{11}$$

$$u_\infty = (2R\theta T_0)^{1/2}\left(\frac{7}{2} + \frac{E_0}{R\theta T_0}\right)^{1/2} \tag{12}$$

and the ratio is

$$\frac{u_{\infty f}}{u_{\infty e}} = \left[1 - \frac{E_F/R\theta}{\frac{7}{2}T_0 + E_0/R\theta}\right]^{1/2} \tag{13}$$

This relation is shown in Fig 3 where $u_{\infty f}/u_\infty$ is plotted vs T_F/T_0 It is found that the curves fall together better when plotted vs T_F/T_0 than they do if they were plotted vs T_F

Assuming that equilibrium exists to the throat, which will be the case in most applications, the mass flux will be the same for both equilibrium and frozen flow Since the equilibrium and frozen flows are considered as having been expanded to the same area, the continuity equation shows that the velocity and density ratios are just the inverse of each other:

$$\rho_{\infty f}/\rho_{\infty e} = (u_{\infty f}/u_\infty)^{-1} \tag{14}$$

Having the density ratio, the temperature and pressure ratios can be found For the frozen expansion from the freeze point where the temperature is T_F and the density is ρ_F, we have the relation

$$\rho_f/\rho_F = (T_f/T_F)^{5/2} \tag{15}$$

For the equilibrium expansion (see Ref 9),

$$\frac{\rho_e}{\rho_F} = \left(\frac{T_e}{T_F}\right)^{5/2}\frac{\exp J_e}{\exp J_F} \tag{16}$$

By dividing Eq (16) by Eq (15) and solving for (T/T_f), we obtain

$$\frac{T_e}{T_f} = \left[\frac{\rho_e}{\rho_f}\frac{\exp J_F}{\exp J}\right]^{2/5} \tag{17}$$

In the high Mach number limit, it can be shown from Eqs (6) and (7) that $\exp J_\infty \to 1$ so that Eq (17) reduces to

$$\frac{T_{\infty e}}{T_{\infty f}} = \left\{\frac{\rho_{\infty e}}{\rho_{\infty f}}\left[\frac{E_F}{R\theta}\exp\frac{1 + (E_F/R\theta)}{T_F}\right]\right\}^{2/5} \tag{18}$$

Since both frozen and equilibrium flow satisfy the same equation of state,

$$\frac{p_{\infty e}}{p_{\infty f}} = \frac{\rho_{\infty e}}{\rho_{\infty f}}\frac{T_{\infty e}}{T_{\infty f}} \tag{19}$$

so that the pressure comparison is

$$\frac{p_{\infty e}}{p_{\infty f}} = \left(\frac{\rho_{\infty e}}{\rho_{\infty f}}\right)^{7/5} \left[\frac{E_F}{R\theta} \exp\frac{1 + (E_F/R\theta)}{T_F}\right]^{2/5} \qquad (20)$$

In Fig 4, this variation of the pressure ratio is plotted as a function of T_F with T_0 as a parameter The curves nearly collapse to a single curve in these coordinates, since most of the variation in Eq (20) comes from the second factor, which is just a function of T_F The density factor is close to one over most of the range

Blunt-Body Flow

In principle, it would be possible to apply Eq (2) to the results of some exact method that gives the flow field around a blunt body, thus finding the surface across which freezing occurs This would be a complicated problem to solve especially in finding the effects of freezing upon the afterbody conditions

In an attempt to simplify the problem as much as possible but still retain the essentials, the following approximation is considered The hypersonic approximation is assumed in which the pressure at the stagnation point of the blunt body is the same independent of the flow process upstream of that point The pressure distribution around the body is the modified Newtonian independent of the flow process along the surface of the body so that $p = p_f$ In addition, the gas is assumed to be in equilibrium at the stagnation point of the body

For detail analysis, a spherically blunted cylinder is selected as a typical geometry Having assumed the pressure distribution to be Newtonian, the other flow variables are obtained from it For this case, Eq (2) can be written

$$\left(\frac{p_0}{p}\frac{\tau'}{\tau_0'}\right)\left(\frac{1}{E}\frac{dE}{dT}\right)\left(p_0\frac{dT}{dp}\right)\left(\frac{D}{p_0}\frac{dp}{dx}\right)\left(\frac{dx/dt}{(2R\theta T_0)^{1/2}}\right) = \frac{D}{\tau_0(2R\theta T_0)^{1/2}} \qquad (21)$$

The term (D/p_0) (dp/dx) can be evaluated from the Newtonian approximation

$$p/p_0 = \cos^2\psi \qquad (22)$$

where ψ is the angle $2x/D$ on the spherical cap

The other factors in Eq (21), except for $p_0(dT/dp)$, are the same functions of temperature as were used in Eq (10) for the analysis of nozzle flow Equation (14) of Ref 9

$$\frac{dE}{R\theta TdT} + \frac{5}{2}\frac{1}{T} - \frac{d\rho}{\rho dT} = 0 \qquad (23)$$

together with the equation of state

$$\frac{dp}{pdT} = \frac{d\rho}{\rho dT} + \frac{1}{T} \qquad (24)$$

yield the remaining term, $p_0(dT/dp)$:

$$p_0\frac{dT}{dp} = \frac{p_0 T}{p}\left(\frac{7}{2} + \frac{dE}{R\theta dT}\right)^{-1} \qquad (25)$$

The blunted-body freezing point criterion of Eq (21) is shown in Fig 5 where $D/[2\tau_0(2R\theta T_0)^{1/2}]$ is plotted against the angle ψ The lines of constant freeze temperature T_F/T_0 are also shown

To find the effects of freezing on afterbody conditions, it is assumed that in both the equilibrium and freezing cases the flow expands to the same afterbody pressure, so that

$$p_f/p_e = 1 = \rho_f T_f/\rho_e T \qquad (26)$$

Using Eq (4) in Eq (26), we obtain

$$\frac{\rho_f}{\rho} = \left(\frac{T_f}{T}\right)^{5/2}\frac{\exp J_e}{\exp J_F} = \left(\frac{T_f}{T}\right)^{-1} \qquad (27)$$

so that solving for T_f/T and assuming the flow has expanded far enough so that $\exp J = 1$ (similar to high Mach number limit in the nozzle case)

$$\frac{T_{\infty f}}{T_\infty} = (\exp J_F)^{2/7} = \frac{\rho_{\infty e}}{\rho_{\infty f}} \qquad (28)$$

Notice that the temperature and density comparison of Eq (28) depend just on $\exp J_F$, which is a function of T_F only Equation (28) is plotted in Fig 6 where the temperature and density comparison ratios are plotted vs T_f The freeze point temperature T_f can be obtained from Fig 5

Concluding Remarks

Equation (1) was introduced as a device to correlate vibrational relaxation data and should be useful for prediction if data is sparse It can also provide a reasonable extrapolation if particular stagnation relaxation data are known The correlation (Fig 1) has the same temperature dependence as is usually found both experimentally and theoretically and requires only the additional parameter τ_θ' to completely

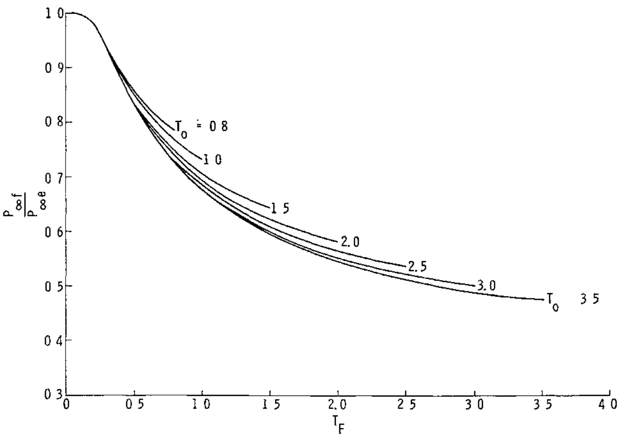

Fig 4 Effect of vibrational freezing on pressure in a nozzle

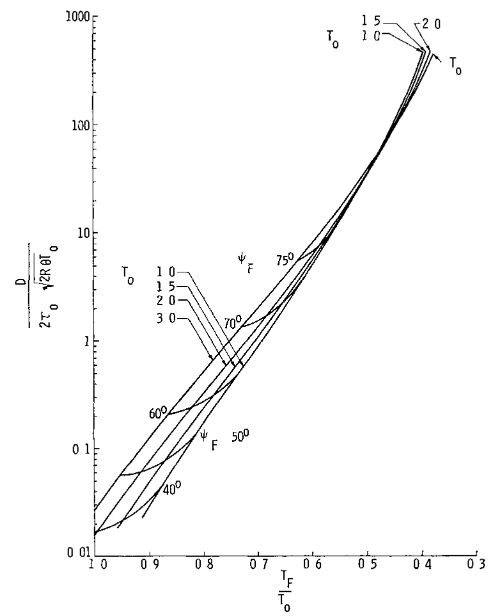

Fig 5 Blunt-body freezing point criterion

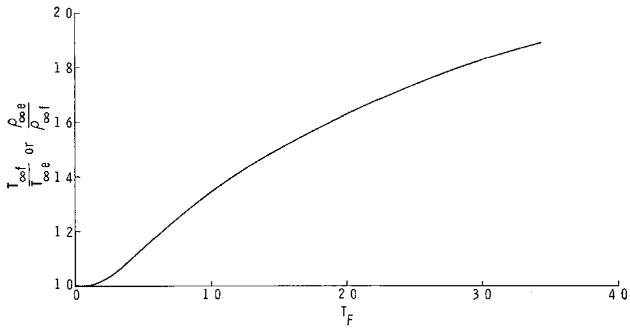

Fig 6 Effect of vibrational freezing on afterbody conditions

characterize the particular gas With this starting point, any problem in vibrational relaxation can be expressed and solved completely with the parameters θ and τ_θ', thus making the results apply to any gas

For nozzle flow, it can be seen from Fig 2 that a major portion of the range of freezing temperature is close to the throat To cover the range of cross sectional area from the throat to $A/A_* = 10$, it is necessary to change τ_0 or r_* by a factor of at least 10^3 In most nozzles in which vibrational freezing occurs, it will be found to take place soon after the throat In Fig 3 it will be observed that the effect of such freezing on the test section velocity is quite small, since the maximum change is of the order of 10%, even if the flow freezes at near stagnation conditions with the temperature as high as 3 5 times the characteristic temperature The effect on test section temperature and pressure is much larger and can be as large as a factor of two Also, note in Fig 4 that the effect of a small amount of frozen energy (small T_F) has a relatively large effect on the test section pressure and temperature

Stagnation temperatures less than 3 5 times the characteristic temperature are likely to be unrealistic due to the onset of dissociation The exact temperature limit depends upon both the particular gas and the pressure in the gas The calculations were extended arbitrarily to 3 5 times the characteristic temperature

A solution to the blunt-body problem with freezing is obtained on the basis of the Newtonian pressure distribution This case also has shown appreciable effects on afterbody temperature and density for large amounts of frozen vibrational energy

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